POSITIVE SIMILARITY SOLUTIONS FOR A DISCRETE VELOCITY BOLTZMANN COAGULATION-FRAGMENTATION MODEL

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Abstract

We consider the Slemrod et al[(1)] coagulation-fragmentation model which is essentially the 2-dimensional Broadwell model including inelastic collisions. We construct two classes of similarity solutions (variable $\eta = x - \zeta t$), positive for $\eta \in (-\infty, \infty)$: the Rankine-Hugoniot solutions and the scalar Riccati similarity solutions. Previous solutions were built up with positivity along half of the x-axis. For the two classes we determine in the parameter space, building up the solutions, domains corresponding to positive solutions.

1. Equations for the Coagulation-Fragmentation Model

In this model[(1)], to the planar Broadwell model with elastic collisions is added inelastic collisions where a cluster with one particle may gain (coagulation with transition probability p_c) or lose (fragmentation with transition probability p_f) only a cluster with one particle. Assuming that the system is symmetric about one $x_1 = x$ axis, we have four independent densities N_i associated to velocities with coordinates x_1, x_2 in the plane.

$N_1 : (1,0), N_2 : (-1,0), N_3 : (0,\pm 1),\ N_0 : (0,0)$

They satisfy two linear relations (mass and momentum conservation laws) and two nonlinear equations: We define: $p_{\pm} = \partial_t \pm \partial_x$, $p_0 = \partial_t$

$$p_+ N_1 + p_0(N_3 + N_0) = 0, p_+ N_1 = p_- N_2, p_+ N_1 = C_1, p_0 N_3 = C_3 \tag{1.1}$$
$$C_1 = [(1-p_c)N_3^2 - (1+p_c)N_1 N_2]/2 + p_f N_0, C_3 = [(1-p_c)N_1 N_2 - (1+p_c)N_3^2]/2 + p_f N_0$$

The elastic collision terms are $\pm(N_1 N_2 - N_3^2)$ and $\pm p_f N_0, \pm p_c(N_1 N_2 + N_3^2)$ for the inelastic collisions. Assuming $N_i(\eta = x - \zeta t)$ we get a system of coupled Riccati equations and the whole study could be done in three steps. First the Rankine-Hugoniot (R-H) relations[(2)] where we consider only the two conservation laws and the vanishing of the collision terms. Second the solutions of the scalar Riccati type[(3)] and third the possible Riccati coupled[(4−3)] solutions. We define two asymptotic states defined by the microscopic densities: (i) n_{0i} and (ii) $s_i = n_{0i} + n_i$, choose $n_{01} = 1$ $n_{02} \geq 1$ and write down the twelve conditions for physical positive solutions:

$$n_{0i} \geq 0, \quad s_i \geq 0, \quad 0 < p_f < 1, \quad 0 < p_c < 1 \tag{1.2}$$

The first problem is whether two positive stable asymptotic states $n_{0i} \geq 0, s_i \geq 0$ exist or not. The best tools for this study are the R-H relations. The second problem is whether or not exist p_c, p_f positive and less than 1. In the R-H framework we will get only p_c/p_f while both p_c and p_f are deduced for the scalar Riccati solutions so that the domain of acceptable solutions will be smaller. We have found two classes of solutions: ClassI with

$0 < \zeta < 1$ for which we will give details and ClassesII with $-1 < \zeta < 0$, deduced from classI by the $\mathcal{T}$ transform: $x \longleftrightarrow -x$, $N_1 \longleftrightarrow N_2$, $N_3 \longleftrightarrow N_3$, $N_0 \longleftrightarrow N_0$.

2. Rankine-Hugoniot Solutions

2.1 General Results

We get six relations. Two from the linear conservation laws where we defined scaled parameters $\overline{n}_j = n_j/n_1$

$$(1-\zeta)n_1 - \zeta(n_3+n_0) = (1-\zeta)n_1 + (1+\zeta)n_2 = 0$$

$$\overline{n}_0 = -[\overline{n}_3 + \overline{n}_2(2+\overline{n}_3)]/(1+\overline{n}_2),\ \zeta = [1+\overline{n}_3+\overline{n}_0]^{-1} = (1+\overline{n}_2)/(1-\overline{n}_2) \tag{2.1}$$

and four from the vanishing of the two collision terms at the asymptotic (i),(ii) states:

$$\text{(i): } n_{00}p_f/p_c = n_{03}^2 = n_{02}n_{01},\ \text{(ii): } s_0p_f/p_c = s_3^2 = s_1s_2 \tag{2.2}$$

or for (ii) $n_0p_f/p_c = n_3^2 + 2n_{03}n_3 = n_2n_{01} + n_1(n_{02}+n_2)$ or

$$n_1 = (\overline{n}_2n_{01} + n_{02} - 2n_{03}\overline{n}_3)/(\overline{n}_3^2 - \overline{n}_2),$$
$$p_f/p_c = (\overline{n}_3^2n_1 + 2n_{03}\overline{n}_3)/\overline{n}_0 = \overline{n}_3[\overline{n}_3(\overline{n}_2n_{01}+n_{02}) - 2n_{03}\overline{n}_2]/\overline{n}_0(\overline{n}_3^2-\overline{n}_2) \tag{2.3}$$

All parameters can be deduced from $n_{01}, n_{02}, \overline{n}_2, \overline{n}_3$. We get successively: n_{03} from (2.2),$\zeta, \overline{n}_0$ from (2.1),$n_1, n_2, n_3, n_0, s_1, s_2, s_3$,$p_c/p_f$ from (2.3-1), n_{00} from (2.2) and finally s_0. The main result is that all s_i can be written in closed form. For the present model the mass $M = N_1 + N_2 + 2(N_3 + N_0)$ and the momentum $J = N_1 - N_2$ are conserved but not the energy and we call m_0, m_s the M values at the (i),(ii) state:

$$s_0 = \overline{n}_0(\sqrt{n_{02}} - \overline{n}_3\sqrt{n_{01}})^2(\overline{n}_3\sqrt{n_{02}} - \overline{n}_2\sqrt{n_{01}})^2/[\overline{n}_3(\overline{n}_3^2 - \overline{n}_2)][\overline{n}_2(\overline{n}_3 - 2n_{03}) + \overline{n}_3n_{02})],$$
$$s_1 = (\overline{n}_3\sqrt{n_{01}} - \sqrt{n_{02}})^2/(\overline{n}_3^2 - \overline{n}_2),\ s_2 = (\overline{n}_3\sqrt{n_{02}} - \overline{n}_2\sqrt{n_{01}})^2/(\overline{n}_3^2 - \overline{n}_2),$$
$$s_3 = (\overline{n}_3\sqrt{n_{02}} - \overline{n}_2\sqrt{n_{01}})(\sqrt{n_{02}} - \overline{n}_3\sqrt{n_{01}})/(\overline{n}_3^2 - \overline{n}_2),\ m_s - m_0 = n_1(1-\overline{n}_2)^2/(1+\overline{n}_2) \tag{2.4}$$

Concerning the stability of the (i),(ii) states we will apply the Lax-Whitham[2] theory. To these (i),(ii) states, assuming that in (2.1-2-3) the n_i are small and the products n_in_j negligible, we associate the characteristic velocities ζ_0, ζ_s of the weak shock theory:

$$\zeta_0^2(n_{01}+n_{02}) + \zeta_0(n_{02}-n_{01}) + (\zeta_0^2-1)2n_{03}/(1+2n_{03}p_c/p_f) = 0, \zeta_0 \to \zeta_s \text{ if } n_{0i} \to s_i \tag{2.5}$$

For ClassI with upstream state (i), the admissibility Lax criterion is $0 < \zeta_0 < \zeta < \zeta_s < 1$. We define the velocity $U = J/M$ with U_0, U_s at the (i),(ii) states, the shock velocities $V_0 = \zeta - U_0, V_s = \zeta - U_s$, sound wave velocities $W_0 = \zeta_0 - U_0$, $W_s = \zeta_s - U_s$ with the shock inequalities $|W| \lessgtr |V|$ at the upstream and downstream states.

2.2 Positivity domain for ClassI

Lemma1 For ClassI, cf. fig.1a, sufficient conditions for positivity of the R-H solutions are

$$n_{01} = 1, n_{02} > 0,\ \sup[-1, -n_{02}] < \overline{n}_2 < 0,\ 0 < \overline{n}_3 < \inf[n_{03}, -2\overline{n}_2/(1+\overline{n}_2)] \tag{2.6}$$

¿From (2.1-2-3-4-6) we get $0 < \zeta < 1$, $\overline{n}_2 < 0$, $\overline{n}_0 > 0$, $s_i > 0$, $p_c/p_f > 0$,$n_{00} > 0$. Only the sign of n_1 is unknown. Depending whether $\overline{n}_3 \lessgtr (\overline{n}_2 + n_{02})/2n_{03}$ we get $n_1 \gtrless 0$, $m_s \gtrless m_0$ with either (i) or (ii) as upstream state and either all $n_i \gtrless 0$ except $n_2 \lessgtr 0$. Only the ratio $\rho = p_c/p_f > 0$ is known and the contour plots are shown in fig.1a for $n_{02} = 2, n_{03} = \sqrt{2}$. We choose $0 < p_c < 1$ such that $p_f = p_c/\rho < 1$.

3. Scalar Riccati Similarity Solutions

3.1 General Results

These solutions, which satisfy the two linear (2.1) relations, are of the type:

$$N_i(\eta) = n_{0i} + n_i/D, D = 1 + w, w = de^{\gamma\eta},\ \eta = x - \zeta t \tag{3.1}$$

The nonlinear (1.1) equations give at the lhs and at the rhs terms of the type:

$$p_+N_1 = (1-\zeta)\gamma n_1(D^{-2} - D^{-1}), p_0N_3 = -\zeta\gamma n_3(D^{-2} - D^{-1})$$

$$N_1N_2 = n_{01}n_{02} + (n_{02}n_1 + n_2n_{01})D^{-1} + n_1n_2D^{-2}, N_3^2 = n_{03}^2 + 2n_3n_{03}D^{-1} + n_3^2D^{-2}$$

Const, D^{-1}, D^{-2} being independent we get six relations: the four R-H (2.2-3) relations and from the coefficients of D^{-2}, two new relations with only one new parameter γ.

$$X_\pm := n_3^2 \pm n_1n_2,\ 2\gamma = (X_- + p_cX_+)/\zeta n_3 = (X_- - p_cX_+)/(1-\zeta)n_1 \tag{3.2}$$

¿From (3.2),(2.3) we rewrite p_c, p_f as functions of $\overline{n}_2, \overline{n}_3$:

$$p_c = -n_0X_-/(2n_3 + n_0)X_+$$

$$p_c = [(\overline{n}_3^2 - \overline{n}_2)/(\overline{n}_3(1+\overline{n}_2) - 2\overline{n}_2)][-\overline{n}_3(1+\overline{n}_2) - 2\overline{n}_2)/(-\overline{n}_3^2 - \overline{n}_2)] \tag{3.3}$$

$$p_f/\overline{n}_3 = (1+\overline{n}_2)(\overline{n}_3(n_{02} + \overline{n}_2n_{01}) - 2\overline{n}_2n_{03})/(\overline{n}_3(1+\overline{n}_2) - 2\overline{n}_2)(-\overline{n}_2 - \overline{n}_3^2) \tag{3.4}$$

3.2 Positivity domain for ClassI

For the positivity properties (1.2) we start with the $\overline{n}_2, \overline{n}_3$ domain (2.6) but now we have a new relation (3.2) which determines p_c and p_f. We must determine the subdomain of the R-H domain for which $0 < p_c < 1, 0 < p_f < 1$. This subdomain is plotted in fig.1b for $n_{02} = 2$ and we see that it is limited by the two curves $p_c = 1$ and $p_f = 1$. We sketch briefly the proof. In (3.3-4) the four factors are positive if we add to (2.6) the condition $\overline{n}_3^2 + \overline{n}_2 < 0$ and it follows $p_c > 0, p_f > 0$.

For the *upper bound* 1 *we discuss first* p_c *which in (3.3) depends only on* $\overline{n}_2, \overline{n}_3$. For $p_c = 1$ we find both $\overline{n}_3 = 0, \overline{n}_2 = 0$ and $\overline{n}_2 = -1 + 2\overline{n}_3 < 0$. Let us consider n_2, n_3 such that: $0 < \overline{n}_3 < \inf\,[\sqrt{-\overline{n}_2},\ -2\overline{n}_2/(1+\overline{n}_2),\ (1+\overline{n}_2)/2]$. The three curves intersect at the same point and it follows, at this stage, that $\overline{n}_3 < \sqrt{2} - 1$. The domain is a triangle limited by the three lines : $\overline{n}_3 = 0\ (p_c = 1) = (1+\overline{n}_2)/2\ (p_c = 1) = -2\overline{n}_2/(1+\overline{n}_2\ (p_c = 0)$. Let $\overline{n}_3 > 0$ fixed, $\overline{n}_2 = -1 + 2\overline{n}_3 + \epsilon$ with ϵ varying so that inside the triangle $0 < \epsilon < 1 - 2\overline{n}_3$ and defining $Y := 1 - 2\overline{n}_3 - \overline{n}_3^2 > 0$ we get $\epsilon < Y/(1+\overline{n}_3/2)$. With $X := (\overline{n}_3 - 1)^2 > 0$ we rewrite the four positive (3.3) factors in terms of $X, Y, \overline{n}_3, \epsilon$:

$$p_c(\epsilon) = (X-\epsilon)(Y - \epsilon(1+\overline{n}_3/2))/(X - \epsilon(1-\overline{n}_3/2))(Y-\epsilon) < p_c(\epsilon = 0) = 1 \tag{3.5}$$

In the triangle, along a line parallel to the $\overline{n}_2$ axis, $p_c > 0$ is decreasing from 1 to 0.

For $p_f/\overline{n}_3$ *which depends on* n_{0i} *and is written in (3.4)*, two of the four factors were present in (3.3). We assume both the (2.6) domain and the restriction inside the above triangle for $0 < p_c < 1$. Still assuming $\overline{n}_3 = (1+\overline{n}_2 - \epsilon)/2$ fixed, $\epsilon > 0$ we define $2Z = \overline{n}_{02}\overline{n}_3 + (1 - 2\overline{n}_3)(2n_{03} - \overline{n}_3) > 0$, notice $Z - \epsilon(n_{03} - \overline{n}_3/2) > \overline{n}_3n_{02}/2$ and get:

$$p_f/\overline{n}_3 = [(2\overline{n}_3 + \epsilon)/(X - \epsilon(1 - \overline{n}_3/2)][Z - \epsilon(n_{03} - \overline{n}_3/2))/(Y - \epsilon)] \tag{3.6}$$

The first bracket increases with ϵ and from $Z - Y(n_{03} - \overline{n}_3/2) = \overline{n}_3(n_{02} + \overline{n}_3(2n_{03} - \overline{n}_3))/2 > 0$ the same property holds for the second factor. For $\overline{n}_3$ fixed, p_f increases with ϵ or with

$\overline{n}_2$. Consequently starting from $\overline{n}_3 = (1+\overline{n}_2)/2$ and letting $\overline{n}_2$ increasing we are limited by the $\overline{n}_2$ value for which $p_f = 1$. In (3.4) we get a quadratic $\overline{n}_2$ polynomial and choose the negative root corresponding to $p_f = 1$.

Lemma2: For the ClassI, in addition to the R-H conditions (2.6), positive scalar Riccati solutions with $0 < p_f < 1, 0 < p_c < 1$ are provided in the $\overline{n}_2, \overline{n}_3$ domain by the curves $\overline{n}_3 < (1+\overline{n}_2)/2$, $(p_c = 1)$ and the $\overline{n}_2 < 0$ root (function of $\overline{n}_3$) given by (3.4) for $p_f = 1$.

In fig.1b, for $n_{02} = 2$, we present the positivity domain, the p_c, p_f contours, find very small values for p_f and only $0.6 < p_c < 1$. As illustration in figs.2a-b for $n_{02} = 2$ we respectively present the N_i and M for $p_c = p_f = 0.8$ and for $p_f = 0.01$, $p_f = 0.99$. In both cases the Lax criterion as well as the shock inequalities are satisfied.

	ζ_0	ζ	ζ_s	V_0	W_0	V_s	W_s	n_{00}	s_0	s_1	s_2	s_3	m_0	m_s
$2a$	0.33	0.55	0.75	0.65	0.43	0.346	0.55	2.	4.3	4.7	0.9	2.1	9.8	18.5
$2b$	0.009	0.08	0.4	0.08	0.01	0.08	0.4	200	213.	2.2	0.96	1.46	404	432

4. ClassII of Positive Solutions

The six R-H (2.1-2) relations remain invariant when we apply the transform:

$\mathcal{T}$: $n_{01} \longleftrightarrow n_{02}$, $n_1 \longleftrightarrow n_2$, $\zeta \longleftrightarrow -\zeta$, $N_1(\eta) \to N_2(-\eta)$, $N_2(\eta) \to N_1(-\eta)$,$N_3(\eta) \to N_3(-\eta)$, $N_0(\eta) \to N_0(-\eta)$, $M(\eta) \to M(-\eta)$

and $n_0, n_{00}, n_3, n_{03}, p_c, p_f$ being unchanged. For the Riccati similarity solutions, the two other relations (3.2) remain invariant under the $\mathcal{T}$ transform if $\gamma \longleftrightarrow -\gamma$. For the exact solutions which depend (Lemmas1-2) on $n_{01}, n_{02}, \overline{n}_2, \overline{n}_3$ they become solutions $n_{02}, n_{01}, 1/\overline{n}_2$, $\overline{n}_3/\overline{n}_2$. As illustration we present in fig2abis the associated solution to fig2a, called[3] "partner solution", and we can verify the mentioned properties.

5. Conclusion

The motivation of this work was to show that in discrete kinetic theory, if we add inelastic collisions, it is still possible to find models with the two asymptotic states which are positive and stable. We find for the Riccati solutions domains smaller than the R-H ones. For the classical[4] Riccati coupled solutions: Projective-Riccati and Conformal-Riccati, we get respectively $p_c = 1$, $\zeta = 0$ which are not possible, but there exists other possibilities[3] that we are investigating.

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